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On Copper Fibonacci and Copper Lucas Sequence

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ABSTRACT

In this paper, we analyzed a new generalization of the Fibonacci sequence that leads to the copper ratio, which we term the copper Fibonacci sequence. Motivated by this definition, we also formulated the corresponding copper Lucas sequence and established several relationships between the terms of these sequences. Various properties were investigated, including explicit Binet-type formulas, special summation identities, and generating functions. Furthermore, we analyzed the roots of the associated characteristic equations and demonstrated how these roots are related to the sequence terms. Finally, we derived important identities for these sequences, including Cassini- and Catalan-type identities.

KEYWORDS

Fibonacci sequence, Lucas Sequence, Binet formula, Copper ratio.

INTRODUCTION

Fibonacci numbers [1-3] form one of the most important sequences in mathematics due to their simple recursive definition and rich theoretical properties.

Metallic ratios [4-6] are encountered across a wide range of scientific disciplines, particularly in the fields of chemistry and biology. Several notable metallic ratios have been identified and are presented in Table I

Number sequences are essential tools for analyzing and understanding the complexity of problems that exhibit patterned behavior. A classical example is the rabbit population problem presented by Leonardo Fibonacci in his renowned work Liber Abaci, which led to the introduction of the Fibonacci sequence and the discovery of the golden ratio as the relationship between its successive terms. Mathematical literature describes several metallic ratios, including the golden, silver, and copper ratios. Among these, the copper ratio appears frequently, particularly in applications related to chemistry and biology, and has therefore gained considerable interest among researchers. In this work, a new sequence, referred to as the copper Fibonacci sequence [7,8], is analyzed based on the copper ratio. This ratio is defined as the quotient of the larger term to the smaller term of two consecutive elements of the sequence. Furthermore, an important property of this sequence is that, under appropriate transformations, its terms can be expressed as single-index terms of the classical Fibonacci sequence.

For $n \in \mathbb{N}$, the Fibonacci numbers F_n and Lucas numbers L_n are defined by the recurrence relations, respectively,

$$F_{n+2} = F_{n+1} + F_n \text{ and } L_{n+2} = L_{n+1} + L_n$$

with the initial conditions $F_0 = 0, F_1 = 1$ and $L_0 = 2, L_1 = 1$.

For F_n and L_n, the Binet formulas are given by the following relations, respectively,

$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $L_n = \alpha^n + \beta^n$,

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are the roots of the characteristic equation $t^2 - t - 1 = 0$. Here, the number α is known as golden ratio.

For $n \in \mathbb{N}$, the bronze Fibonacci numbers BF_n, bronze Lucas numbers BL_n, third-order bronze Fibonacci numbers Bf_n, third-order bronze Lucas numbers Bl_n, and modified third-order bronze Fibonacci numbers MBf_n are defined by the recurrence relations, respectively,

$$BF_{n+2} = 3BF_{n+1} + BF_n, BL_{n+2} = 3BL_{n+1} + BL_n,$$

$$Bf_{n+3} = 3Bf_{n+2} + Bf_{n+1} + Bf_n, \quad Bl_{n+3} = 3Bl_{n+2} + Bl_{n+1} + Bl_n,$$

and

$$MBf_{n+3} = 3MBf_{n+2} + MBf_{n+1} + MBf_n,$$

with the initial conditions $BF_0 = 0$, $BF_1 = 1$, $BL_0 = 2$, $BL_1 = 3$, $Bf_0 = 0$, $Bf_1 = 1$, $Bf_2 = 3$, $Bl_0 = 3$, $Bl_1 = 3$ and $Bl_2 = 11$ and $MBf_0 = 1$, $MBf_1 = 2$, $MBf_2 = 7$.

For BF_n and BL_n, the Binet formulas are given, respectively, by the following relationships:

$$BF_n = \frac{\lambda^n - \psi^n}{\lambda - \psi} \text{ and } BL_n = \lambda^n + \psi^n,$$

where $\lambda = \frac{3+\sqrt{13}}{2}$ and $\psi = \frac{3-\sqrt{13}}{2}$ are the roots of the characteristic equation $r^2 - 3r - 1 = 0$. Here, the λ number is the known bronze ratio.

Quaternions were first described by Hamilton in 1843. In addition, quaternions are used to control rotational movements, especially in kinematics, 3D games, mechanics, Eulerian angles, and chemistry. Horadam defined complex Fibonacci and Fibonacci quaternions, and various features were found.

The algebra of hyperbolic quaternions is an algebra that is not related to the elements of the form over the real numbers,

$$q = xi_1 + yi_2 + zi_3 + ti_4, x, y, z, t \in R.$$

TABLE I: Metallic ratios

Ratio	Value	OEIS
Plastic	1.324 717 957 244 7...	A182097
Gold	1.618033988749 895...	A001622
Silver	2.414213562373 095...	A014176
Bronze	3.302775637731 995...	A098316
Copper	4.236067 977 499 79...	A098317

Nickel	5.192582403567 252...	A098318
Aluminum	6.162277 660 168 38...	A176398
Iron	7.140054944640 259...	A176439
Tin	8.123105625617 661...	A176458
Lead	9.109772228646 444...	A176522

In this paper, we define the copper Fibonacci sequences CF_n, and the copper Lucas sequence CL_n. We give the generating functions, summation formulas, the Binet formulas, and some properties for these sequences.

II Copper Fibonacci and Copper Lucas sequences

If one removes four largest possible squares from a rectangle with the ratio length/width equal to $2+\sqrt{5}$, one gets a rectangle with the same ratio length/width. Now let us examine following Figure 1.1.

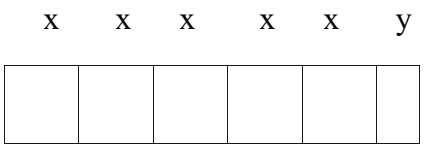

Figure 1.1 Copper rectangle

According to the rectangle, we obtain

$$\frac{4x+y}{x} = \frac{x}{y} \text{ or } 4+\frac{y}{x} = \frac{x}{y}.$$

Thus, we find

$$\frac{x}{y} = 2+\sqrt{5}.$$

At the same time, when we remove these squares from Figure 1.1 and continue the process indefinitely, we find the same ratio, that is, the copper ratio. We can say that the series we define shortly will be used in many areas of mathematics. Applications in groups, coding theory, and applications in quaternions can be given as examples.

Definition 1.1

For $n \in \mathbb{N}$, the copper Fibonacci sequence CF_n and copper Lucas sequence CL_n are defined, respectively, as

$$CF_{n+2} = 4CF_{n+1} + CF_n, \tag{1.1}$$

with $CF_0 = 0$ and $CF_1 = 1$, and

$$CL_{n+2} = 4CL_{n+1} + CL_n, \tag{1.2}$$

with $CL_0 = 2$ and $CL_1 = 4$.

Then let us give some information about the equations of these sequence is

$$r^2 - 4r - 1 = 0. \tag{1.3}$$

The roots of the characteristic equation are as follows:

$\kappa = 2 + \sqrt{5} \cong 4.23606797749979 \ldots,$ and $\theta = 2 - \sqrt{5}$.

Here, the κ number is the known copper ratio.

Next, we give the relationships between these roots below:

$$\kappa + \theta = 4, \kappa - \theta = 2\sqrt{5}, \kappa^2 + \theta^2 = 18, and\ \kappa\theta = -1.$$

The first few values of copper Fibonacci and copper Lucas sequence are

$$0, 1, 4, 17, 72, 305, 1292, 5473, 23184, 98209, \ldots.$$

and

$$2, 4, 18, 76, 322, 1364, 5778, 24476, 103682, \ldots .,$$

respectively.

In addition, the terms of the copper Fibonacci and copper Lucas sequence can be found with the help of the following relations. Let $n \in N^+$,

$$CF_n = \sum_{q=0}^{\frac{n-1}{2}} \binom{n-1-q}{q} 4^{n-1-2q} \text{ and } CL_n = \sum_{q=0}^{\frac{n}{2}} \frac{n}{n-q} \binom{n-1-q}{q} 4^{n-2q}.$$

In the following theorem, we express the Binet formulas of the copper Fibonacci sequence CF_n and copper Lucas sequence CL_n.

Theorem 1.1

Let $n \in \mathbb{N}$. We obtain

(i) $CF_n = \frac{\kappa^n - \theta^n}{\kappa - \theta}$ and

(ii) $CL_n = \kappa^n - \theta^n$.

Proof

(i) The Binet form of a sequence is as follows:

$$CF_n = S\kappa^n + T\theta^n.$$

The scalars S and T can be obtained by substituting the initial conditions and solving the given system of equations.

For $n = 0$, $CF_0 = 0$, and for $n = 1$, $CF_1 = 1$.

Hence, $S = \frac{1}{\kappa-\theta}$ and $T = \frac{-1}{\kappa-\theta}$.

Thus, we obtain

$$CF_n = \frac{\kappa^n - \theta^n}{\kappa - \theta}$$

(ii) The Binet form of a sequence is as follows:

$$CL_n = S\kappa^n + T\theta^n.$$

The scalars S and T can be obtained by substituting the initial conditions and solving the given system of equations.

For $n = 0$, $CL_0 = 2$, and for $n = 1$, $CL_1 = 4$.

Hence, $S = 1$ and $T = 1$.

Thus, we obtain

$$CL_n = \kappa^n - \theta^n$$

Theorem 1.2

The ratio of the larger to the smaller of two consecutive terms of the copper Fibonacci and copper Lucas sequence converges to the copper ratio. We obtain

(i) $\lim_{n\to\infty} \frac{CF_{n+1}}{CF_n} = \kappa$ and

(ii) $\lim_{n\to\infty} \frac{CL_{n+1}}{L} = \kappa$

Proof

(i) If the Binet formula is used, we have

$$\lim_{n\to\infty}\frac{CF_{n+1}}{CF_n}=\frac{\frac{\kappa^{n+1}-\theta^{n+1}}{\kappa-\theta}}{\frac{\kappa^{n}-\theta^{n}}{\kappa-\theta}}$$

$$=\lim_{n\to\infty}\frac{\kappa^{n+1}\left(1-\left(\frac{\theta}{\kappa}\right)^{n+1}\right)}{\kappa^{n}\left(1-\left(\frac{\theta}{\kappa}\right)^{n}\right)}$$

Thus, we obtain

$$\lim_{n\to\infty}\frac{CF_{n+1}}{CF_n}=\kappa$$

(ii) If the Binet formula is used, we have

$$\lim_{n\to\infty}\frac{CL_{n+1}}{CL_n}=\frac{\kappa^{n+1}-\theta^{n+1}}{\kappa^{n}-\theta^{n}}$$

Thus, we obtain

$$\lim_{n\to\infty}\frac{CL_{n+1}}{CL_n}=\kappa$$

Theorem 1.3

Let $m,n,p,r\in\mathbb{N}$. We obtain

(i) $$\sum_{m=0}^{n}CF_{rm+p}=\begin{cases}\frac{-CF_{nr+n+p}+(-1)^{r}CF_{nr+p}+CF_{r-p}+CF_{p}}{1+(-1)^{r}-CL_{r}}, if\ p<r,\\ \frac{-CF_{nr+n+p}+(-1)^{r}CF_{nr+p}+CF_{p-r}+CF_{p}}{1+(-1)^{r}-CL_{r}}, otherwise,\end{cases}$$

and

(ii) $$\sum_{m=0}^{n}CL_{rm+p}=\begin{cases}\frac{-CL_{nr+n+p}+(-1)^{r}CL_{nr+p}-CL_{r-p}+CL_{p}}{1+(-1)^{r}-CL_{r}}, if\ p<r,\\ \frac{(-1)^{r}CL_{nr+p}-CL_{p-r}+CL_{p}-CF_{nr+n+p}}{1+(-1)^{r}-CL_{r}}, otherwise.\end{cases}$$

Proof

(i) If the Binet formulas are used, we get

$$\sum_{m=0}^{n}CF_{rm+p}=\sum_{y=0}^{n}\frac{\kappa^{rm+p}-\theta^{rm+p}}{\kappa-\theta}$$

$$=\frac{\kappa^{p}}{\kappa-\theta}\sum_{y=0}^{n}(\kappa^{r})^{m}-\frac{\theta^{p}}{\kappa-\theta}\sum_{y=0}^{n}(\theta^{r})^{m}$$

$$= \frac{\kappa^p}{\kappa-\theta}\frac{\kappa^{nr+1}-1}{\kappa^p-1} - \frac{\theta^p}{\kappa-\theta}\frac{\theta^{nr+1}-1}{\theta^p-1}$$

$$\sum_{m=0}^{n} CF_{rm+p} = \begin{cases} \dfrac{-CF_{nr+n+p} + (-1)^r CF_{nr+p} + CF_{r-p} + CF_p}{1+(-1)^r - CL_r}, if\ p < r, \\ \dfrac{-CF_{nr+n+p} + (-1)^r CF_{nr+p} + CF_{p-r} + CF_p}{1+(-1)^r - CL_r}, otherwise, \end{cases}$$

(ii) If the Binet formulas are used, we get

$$\sum_{m=0}^{n} CL_{rm+p} = \sum_{y=0}^{n} \kappa^{rm+p} - \theta^{rm+p}$$

$$\sum_{m=0}^{n} CL_{rm+p} = \begin{cases} \dfrac{-CL_{nr+n+p} + (-1)^r CL_{nr+p} - CL_{r-p} + CL_p}{1+(-1)^r - CL_r}, if\ p < r, \\ \dfrac{(-1)^r CL_{nr+p} - CL_{p-r} + CL_p - CF_{nr+n+p}}{1+(-1)^r - CL_r}, otherwise. \end{cases}$$

Corollary 1.1

The sum of the consecutive, odd, and even indexed terms of the copper Fibonacci and copper Lucas sequences are as follows,

(i) $\sum_{m=0}^{n} CF_m = \frac{5CF_n + CF_{n-1} - 1}{4}$,

(ii) $\sum_{m=0}^{n} CL_m = \frac{5CL_n + CL_{n-1} + 2}{4}$,

(iii) $\sum_{m=0}^{n} CF_{2m} = \frac{CF_{2n+2} - CF_{2n} - CF_2}{16}$,

(iv) $\sum_{m=0}^{n} CL_{2m} = \frac{CL_{2n+2} - CL_{2n} - CL_2 - 2}{16}$,

(v) $\sum_{m=0}^{n} CF_{2m+1} = \frac{CF_{2n+3} - CF_{2n+1} - 2}{16}$, and

(vi) $\sum_{m=0}^{n} CL_{2m+1} = \frac{CL_{2n+3} - CL_{2n+1}}{16}$.

Theorem 1.4

Let $r, n, p \in \mathbb{N}$ and $p > r$; we obtain

(i) $\sum_{n=0}^{\infty} CF_{rn} + px^n = \frac{CF_p - x(-1)^r CF_{p-r}}{1 - xCL_r + (-1)^r x^2}$,

(ii) $\sum_{n=0}^{\infty} CL_{rn} + px^n = \frac{CL_p - x(-1)^r CL_{p-r}}{1 - xCL_r + (-1)^r x^2}$,

(iii) $\sum_{n=0}^{\infty} \frac{CF_{rn}}{n!} x^n = \frac{e^{\kappa^r x} - e^{\theta^r x}}{\kappa - \theta}$, and

(iv) $\sum_{n=0}^{\infty} \frac{CL_{rn}}{n!} x^n = e^{\kappa^r x} - e^{\theta^r x}$.

Proof

(i) If the Binet formula is used, we have

$$\sum_{n=0}^{\infty} CF_{rn} + px^n = \sum_{n=0}^{\infty} \frac{\kappa^{rn+p} - \theta^{rn+p}}{\kappa - \theta} x^n$$

$$= \frac{\kappa^p}{\kappa - \theta} \sum_{n=0}^{\infty} (\kappa^r x)^n - \frac{\theta^p}{\kappa - \theta} \sum_{n=0}^{\infty} (\theta^r x)^n$$

$$= \frac{\kappa^p}{\kappa - \theta} \frac{1}{1 - \kappa^r x} - \frac{\theta^p}{\kappa - \theta} \frac{1}{1 - \theta^r x}$$

$$\sum_{n=0}^{\infty} CF_{rn} + px^n = \frac{CF_p - x(-1)^r CF_{p-r}}{1 - xCL_r + (-1)^r x^2}$$

(ii) If the Binet formula is used, we have

$$\sum_{n=0}^{\infty} CL_{rn} + px^n = \sum_{n=0}^{\infty} \kappa^{rn+p} - \theta^{rn+p} \, x^n$$

$$\sum_{n=0}^{\infty} CF_{rn} + px^n = \frac{CF_p - x(-1)^r CF_{p-r}}{1 - xCL_r + (-1)^r x^2}$$

(iii) If the Binet formula is used, we have

$$\sum_{n=0}^{\infty} \frac{CF_{rn}}{n!} x^n = \sum_{n=0}^{\infty} \frac{\kappa^{rn} - \theta^{rn}}{\frac{\kappa^r - \theta r^n}{n!}} x^n$$

$$= \frac{1}{\kappa - \theta} \sum_{n=0}^{\infty} \frac{(\kappa^r x)^n}{n!} - \frac{1}{\kappa - \theta} \sum_{n=0}^{\infty} \frac{(\theta^r x)^n}{n!}$$

$$\sum_{n=0}^{\infty} \frac{CF_{rn}}{n!} x^n = \frac{e^{\kappa^r x} - e^{\theta^r x}}{\kappa - \theta}$$

(iv) If the Binet formula is used, we have

$$\sum_{n=0}^{\infty} \frac{CL_{rn}}{n!} x^n = \sum_{n=0}^{\infty} \frac{\kappa^{rn} + \theta^{rn}}{n!} x^n$$

$$= \sum_{n=0}^{\infty} \frac{(\kappa^r x)^n}{n!} + \sum_{n=0}^{\infty} \frac{(\theta^r x)^n}{n!}$$

$$\sum_{n=0}^{\infty} \frac{CL_{rn}}{n!} x^n = e^{\kappa^r x} - e^{\theta^r x}$$

Corollary 1.2

The generating functions of copper Fibonacci and copper Lucas sequences are as follows:

(i) $f(x) = \sum_{n=0}^{\infty} CF_n x^n = \frac{x}{1-4x-x^2}$

(ii) $l(x) = \sum_{n=0}^{\infty} CL_n x^n = \frac{2-4x}{1-4x-x^2}$.

Theorem 1.5 (Cassini identity)

Let $n \in \mathbb{N}$. We obtain

(i) $CF_{n+1}CF_{n-1} - CF_n^2 = (-1)^n \; and$

(ii) $CL_{n+1}CL_{n-1} - CL_n^2 = 20(-1)^{n-1}$.

Proof

If the Binet formulas are used, we obtain

(i) $$CF_{n+1}CF_{n-1} - CF_n^2 = \frac{k^{n+1}-\theta^{n+1}}{k-\theta}\frac{k^{n-1}-\theta^{n-1}}{k-\theta} - \frac{k^n-\theta^n}{k-\theta}\frac{k^n-\theta^n}{k-\theta}$$

$$= \frac{k^{2n}-k^{n+1}\theta^{n-1}-\theta^{n+1}k^{n-1}+\theta^{2n}}{(k-\theta)^2} - \frac{k^{2n}-2k^n\theta^n+\theta^{2n}}{(k-\theta)^2}$$

$$= \frac{(k\theta)^n\frac{-k}{\theta}}{(k-\theta)^2} + \frac{(k\theta)^n\frac{-\theta}{k}}{(k-\theta)^2} + \frac{2(k\theta)^n}{(k-\theta)^2}$$

$$= (-1)^n$$

and

(ii) $$CL_{n+1}CL_{n-1} - CL_n^2 = (k^{n+1} + \theta^{n+1})(k^{n-1} + \theta^{n-1}) - (k^n + \theta^n)(k^n + \theta^n)$$

$$= k^{2n} + k^{n+1}\theta^{n-1} + \theta^{n+1}k^{n-1} + \theta^{2n}k^{n-1} + \theta^{2n} - k^{2n} - 2k^n\theta^n - \theta^{2n}$$

$= (k\theta)^n(-\frac{k}{\theta} - \frac{\theta}{k} + 2)$

$= 20(-1)^{n-1}.$

Theorem 1.6 (Catalan identity)

Let $n,\ i \in \mathbb{N}$ and $i < n$. We obtain

(i) $CF_{n+i}CF_{n-i} - {CF_n}^2 = (-1)^{n-i+1}{CF_i}^2$ and

(ii) $CL_{n+i}CL_{n-i} - {CL_n}^2 = 20(-1)^{n-i}{CF_i}^2$

Theorem 1.7 (D'ocagne identity)

Let $n,\ i \in \mathbb{N}$ and $i < n$. We obtain

(i) $CF_{n+1}CF_i - CF_nCF_{i+1} = (-1)^{i+1}CF_{n-1}$ and

(ii) $CL_{n+1}CL_i - CL_nCL_{i+1} = 20(-1)^iCF_{n-i}.$

Theorem 1.8 (Vajda identity)

Let $n,\ i,j \in \mathbb{N}$ and $i,j < n$. We obtain

(i) $CF_{n+i}CF_{n+j} - CF_nCF_{n+i+j} = (-1)^nCF_iCF_j$ and

(ii) $CL_{n+i}CL_{n+j} - CL_nCL_{n+i+j} = 20(-1)^{n+1}CF_iCF_j$

Theorem 1.9 (Halton identity)

Let $n,\ i,j \in \mathbb{N}$ and $i,j < n$. We obtain

(i) $CF_{n+i}CF_{n-i} - CF_{n+j}CF_{n-j} = \frac{1}{20}\left[(-1)^{n-j}BL_{2j} - (-1)^{n-i}BL_{2i}\right]$

(ii) $BL_{n+i}BL_{n-i} - BL_{n+j}BL_{n-j} = \left[(-1)^{n-i}BL_{2i} - (-1)^{n-j}BL_{2j}\right]$

Theorem 1.10 (Padilla identity)

Let $\in \mathbb{N}$. We obtain

(i) ${CF_{n+2}}^3 + {CF_{n-1}}^3 - 3CF_nCF_{n+1}CF_{n+2} = \frac{1}{20}(CF_{3n+6} - 3CF_{3n+3} + CF_{3n-3} + 3CF_{2n+3} - 3CF_{n+2} + 3CF_{n+1})$ and

(ii) ${CL_{n+2}}^3 + {CL_{n-1}}^3 - 3CL_nCL_{n+1}CL_{n+2} = (CL_{3n+6} + 3CL_{n+2} + CL_{3n-3} + 3CL_{n-1} + 20)(-CF_{3n+3} - 3CF_{n-1} + 3CF_{n+1} + 3CF_{2n+3})$

Theorem 1.10 (Melham identity)

Let $\in \mathbb{N}$. We obtain

(i) $CF_{n+1}CF_{n+2}CF_{n+6} - {CF_n}^3 = \frac{1}{20}(CF_{3n+9} - CF_{3n} - \frac{1}{(-1)^{n+1}}CF_{n+7} - \frac{1}{(-1)^{n+2}}CF_{n+5} + \frac{3}{(-1)^n}CF_n - \frac{1}{(-1)^{n+6}}CF_{n-3}$ and

(ii) $CL_{n+1}CL_{n+2}CL_{n+6} - {CL_n}^3 = (CL_{3n+9} - CL_{3n} - \frac{1}{(-1)^{n+1}}CL_{k,n+7} - \frac{1}{(-1)^{n+2}}CL_{k,n+5} + \frac{3}{(-1)^n}CL_n - \frac{1}{(-1)^{n+6}}CL_{n-3}$

Theorem 2.12 (Gelin-Cesaro identity)

Let $n \in \mathbb{N}$. We obtain

(i) $CF_{n+2}CF_{n+1}CF_{n-1}CF_{n-2} - CF_n^4 = \frac{1}{20}(\frac{1}{(-1)^{n-3}}CL_{2n+4} + \frac{1}{(-1)^{n-2}}CL_{2n+2} + \frac{1}{(-1)^{2n-3}}CL_6 + \frac{1}{(-1)^n}CL_{2n-2} + CL_2 + \frac{1}{(-1)^{n+1}}CL_{2n-4} + \frac{1}{(-1)^{n-3}}CL_{2n} - 90\ and$

(ii) $CL_{n+2}CL_{n+1}CL_{n-1}CL_{n-2} - CL_n^4$
$= \frac{1}{(-1)^{n-2}}CL_{2n+4} + \frac{1}{(-1)^{n-1}}CL_{2n+2} + \frac{1}{(-1)^{2n-3}}CL_6 + \frac{1}{(-1)^{n+1}}CL_{2n-2} + + \frac{1}{(-1)^{2n-1}}18 + \frac{1}{(-1)^{n+2}}CL_{2n-4} - \frac{1}{(-1)^{n-2}}CL_{2n-5}.$

CONCLUSION

In this work, we examined a novel extension of the Fibonacci sequence which we refer to as the copper Fibonacci sequence that results in the copper ratio. Inspired by this description, we generated the analogous Lucas sequence for copper and determined a number of connections between these sequences' terms. Special summation identities, generating functions, and explicit Binet-type formulas were among the properties that were examined. Additionally, we examined the roots of the corresponding characteristic equations and showed how the sequence terms connect to these roots. Lastly, we deduced significant identities for these sequences, such as identities of the Cassini and Catalan types.

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